Critical phenomena in higher curvature charged AdS black holes

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Abstract

In this paper we have studied the critical phenomena in higher curvature charged black holes in the anti-de Sitter (AdS) space-time. As an example we have considered the third order Lovelock-Born-Infeld black holes in AdS space-time. We have analytically derived the thermodynamic quantities of the system. Our analysis revealed the onset of a higher order phase transition in the black hole leading to an infinite discontinuity in the specific heat at constant charge at the critical points. Our entire analysis is based on the canonical framework where we have fixed the charge of the black hole. In an attempt to study the behavior of the thermodynamic quantities near the critical points we have derived the critical exponents of the system explicitly. Although the values of the critical points have been determined numerically, the critical exponents are calculated analytically. Our results fit well with the thermodynamic scaling laws. The scaling hypothesis is also seen to be consistent with these scaling laws. We find that all types of AdS black holes, studied so far, indeed belong to the same universality class. Moreover these results are consistent with the mean field theory approximation. We have derived the suggestive values of the other two critical exponents associated with the correlation function and correlation length on the critical surface.

1 Introduction

String theory and brane world cosmology underscored the necessity of considering space-time as multidimensional. This motivates us to study gravity theories in higher dimensions (i.e. greater than four)[1]. The effect of string theory on gravity may be understood by considering a low energy effective action that describes the classical gravity[2]. This effective action must include combinations of the higher curvature terms and are found to be ghost free[3]. In an attempt to obtain the most general tensor that satisfies the properties of Einstein's tensor in higher dimensions, Lovelock proposed an effective action that contains higher curvature terms[4]. Moreover, the field equations derived from this action consists of only second derivatives of the metric and hence are free of ghosts[5]. The study of black holes in higher dimensions has found renewed attention because of the enriched physics associated with them[6]-[8].

Thermodynamics of black holes has been a principal topic of research due to the discovery of the laws of black hole mechanics[9]. The study of thermodynamic properties of black holes in anti-de Sitter space-time has got renewed attention due to the discovery of phase transition in Schwarzschild-AdS black holes[10]. Since then a wide variety of research has been done to study the phase transition in black holes[7, 8], [11]-[22]. Now a days, the study of thermodynamics of black holes in AdS space-time is very much important in the context of AdS/CFT duality. The thermodynamics of AdS black holes may en-light the underlying phase structure and thermodynamic properties of CFTs[23]. The study of thermodynamic phenomena in black holes requires an analogy between the variables in ordinary thermodynamics and those in black

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hole mechanics. Recently, R. Banerjee *et. al.* have developed a method[18, 19], based on Ehrenfest's scheme[24] of standard thermodynamics, to study the phase transition phenomena in black holes. In this approach one can actually determine the order of phase transition once the relevant thermodynamic variables are identified for the black holes. This method has been successfully applied in the four dimensional black holes in AdS space-time[19]-[22] as well as in higher dimensional AdS black holes[7]. While most of the black holes undergo second order phase transitions characterized by an infinite discontinuity in the specific heat at the critical points (i.e. points of divergence), one exception is observed in the charged Hořava-Lifshitz black holes in AdS space[25]. Here we observe both first and second order phase transitions.

The behavior of the thermodynamic variables near the critical points can be studied by means of a set of indices known as static critical exponents[26, 27]. These exponents are to a large extent universal, independent of the spatial dimensionality of the system and obey thermodynamic scaling laws[26, 27]. The critical phenomena has been studied extensively in familiar physical systems like the Ising model (two and three dimensions), magnetic systems, elementary particles, hydrodynamic systems etc.. An attempt to study the critical phenomena in black holes was commenced in the last twenty years[28]-[43]. Despite all these attempts, a systematic study of critical phenomena in black holes was still lacking. This problem has been overcome very recently[8, 44]. In these works the critical phenomena have been studied in (3+1) as well as higher dimensional AdS black holes. Also the critical exponents of the black holes have been determined by explicit analytic calculations.

All the research mentioned above was confined to Einstein gravity. It would then be interesting to study gravity theories in which action involves higher curvature terms. Among the higher curvature black holes, Gauss-Bonnet and Lovelock black holes will be suitable candidates to study. The thermodynamic properties and phase transitions have been studied in the Gauss-Bonnet AdS (GB-AdS) black holes[45]-[49]. Also, the critical phenomena in the GB-AdS black holes was studied[50]. On the other hand, Lovelock gravity coupled to the Maxwell field was investigated in [51, 52]. The thermodynamic properties of the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black holes were studied in [51], [53, 54]. But the study of phase transition and critical phenomena have not been done yet in these black holes.

In this paper we have studied the critical phenomena in the third order LBI-AdS black holes in seven dimensions. We also have given a qualitative discussion about the possibility of higher order phase transition in this type of black holes. We have determined the static critical exponents for these black holes and showed that these exponents obey the static scaling laws. We have also checked the static scaling hypothesis and calculated the scaling parameters. We see that these critical exponents take the mean field values. From our study of the critical phenomena we may infer that the third order LBI-AdS black holes yield results consistent with the mean field theory approximation. There are some distinct features in the higher curvature gravity theory. For example, the usual area law valid in Einstein gravity does not hold in these gravity theories. However, the critical exponents are identical with those in the usual Einstein gravity. This result shows that the AdS black holes belong to the same universality class. We have also determined the critical exponents associated with the correlation function and correlation length. However the values of these exponents are more suggestive than definitive. As a final remark, we have given a qualitative argument for the determination of these last two exponents.

Let us mention about the outline of the paper. In section 2 we discuss about the thermodynamical variables of the seven dimensional third order LBI-AdS black holes. In section 3 we analyze the phase transition and stability of these black holes. The critical exponents, scaling laws and scaling hypothesis are discussed in section 4. Finally, we have drawn our conclusions in section 5.

2 Thermodynamic variables of higher curvature charged AdS black holes

The effective action in the Lovelock gravity in $(n+1)$ dimensions can be written as[1][4]

$$\mathcal{I} = \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\sum_{i=0}^{[\frac{n+1}{2}]}\alpha_i\mathcal{L}_i \tag{1}$$

where α_i is an arbitrary constant and $\mathcal{L}_i$ is the *Euler density* of a $2i$ dimensional manifold. In $(n+1)$ dimensions all terms for which $i > [(n+1)/2]$ are equal to zero, the term $i = (n+1)/2$ is a topological term and terms for which $i < [(n+1)/2]$ contribute to the field equations. Since we are studying third order Lovelock gravity in the presence of Born-Infeld nonlinear electrodynamics[55] the effective action (1) may be written as

$$\begin{aligned}\mathcal{I} &= \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left(\alpha_0\mathcal{L}_0+\alpha_1\mathcal{L}_1+\alpha_2\mathcal{L}_2+\alpha_3\mathcal{L}_3+L(F)\right)\\ &= \frac{1}{16\pi}\int d^{n+1}x\sqrt{-g}\left(-2\Lambda+\mathcal{R}+\alpha_2\mathcal{L}_2+\alpha_3\mathcal{L}_3+L(F)\right)\end{aligned} \tag{2}$$

where Λ is the cosmological constant given by $-n(n-1)/2l^2$, l being the string length, α_2 and α_3 are the second and third order Lovelock coefficients, $\mathcal{L}_1 = \mathcal{R}$ is the usual Einstein-Hilbert Lagrangian, $\mathcal{L}_2 = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + \mathcal{R}^2$ is the Gauss-Bonnet Lagrangian and

$$\begin{aligned}\mathcal{L}_3 = 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu} + 8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}{}_{\mu} +\\ 3\mathcal{R}R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu} - 12\mathcal{R}R^{\mu\nu}R_{\mu\nu} + \mathcal{R}^3\end{aligned}$$

is the third order Lovelock Lagrangian, $L(F)$ is the Born-Infeld Lagrangian given by

$$L(F) = 4b^2\left(1-\sqrt{1+\frac{F^2}{2b^2}}\right) \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^2 = F_{\mu\nu}F^{\mu\nu}$ and b is the Born-Infeld parameter. In the limit $b \to \infty$ we recover the standard Maxwell form $L(F) = -F^2$.

The solution of the third order Lovelock-Born-Infeld anti de-Sitter black hole (LBI-AdS) in $(n+1)$-dimensions can be written as[54],

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{k,n-1} \tag{4}$$

where $d\Omega^2_{k,n-1} = \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1}\prod_{j=1}^{i-1} sin^2\theta_j d\theta_i^2 & k=1\\ d\theta_1^2 + sinh^2\theta_1 d\theta_2^2 + sinh^2\theta_1\sum_{i=3}^{n-1}\prod_{j=2}^{i-1} sin^2\theta_j d\theta_i^2 & k=-1\\ \sum_{i=1}^{n-1} d\phi_i^2 & k=0\end{cases}$

and k determines the structure of the black hole horizon[2].

Since we are considering third order LBI-AdS black hole, we shall restrict ourselves in the seven dimensional space-time $(n+1=7)$. Moreover, the Lagrangian of (2) is the most general Lagrangian in seven space-time dimensions that produces the second order field equations[51].

[1] Here we have taken the gravitational constant $G = 1$.

[2] $k = +1(spherical), -1(hyperbolic), 0(planar)$

One can show that in the special case $\alpha_3 = 2\alpha_2^2 = \frac{\alpha^2}{72}$, the metric function $f(r)$ of (4) may be expressed as[54],

$$f(r) = k + \frac{r^2}{\alpha}\left(1 - \chi(r)^{\frac{1}{3}}\right) \tag{5}$$

where

$$\chi(r) = 1 + \frac{3\alpha m}{r^6} - \frac{2\alpha b^2}{5}\left[1 - \sqrt{1+\eta} - \frac{\Lambda}{2b^2} + \frac{5\eta}{4}\mathcal{H}(\eta)\right] \tag{6}$$

In (6)

$$\eta = \frac{10q^2}{b^2 r^{10}} \tag{7}$$

and $\mathcal{H}(\eta)$ is the hyper geometric function given by[56],

$$\mathcal{H}(\eta) = \mathcal{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\eta\right), \tag{8}$$

q and m are constants which are related to the charge Q and the ADM mass M of of the black hole through the equations

$$Q = \frac{\mathcal{V}_{n-1}}{4\pi}\sqrt{\frac{(n-1)(n-2)}{2}}q = \frac{\sqrt{10}\pi^2 q}{4} \tag{9}$$

and

$$M = \frac{\mathcal{V}_{n-1}}{16\pi}(n-1)m = \frac{5\pi^2}{16}m \tag{10}$$

respectively. Here $\mathcal{V}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the volume of the $(n-1)$ sphere.

The ADM mass (M) of the third order LBI-AdS black hole may be defined as $f(r_+) = 0$, which yields,

$$M = \frac{5\pi^2}{16}\left[\frac{\alpha^2}{3} + r_+^4 + \alpha r_+^2 + \frac{2b^2 r_+^6}{15}\left(1 - \sqrt{1+\eta_+} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathcal{H}(\eta_+)\right)\right] \tag{11}$$

The electrostatic potential difference between the black hole horizon and the infinity may be defined as[54],

$$\Phi = \sqrt{\frac{(n-1)}{2(n-2)}}\frac{q}{r_+^{n-2}}\mathcal{H}(\eta_+) = \frac{Q}{\pi^2 r_+^4}\mathcal{H}(\eta_+) \tag{12}$$

where

$$\eta_+ = \frac{16Q^2}{b^2\pi^4 r_+^{10}}. \tag{13}$$

Here we have used (9).

Using the metric function (5) one may find the Hawking temperature for the third order LBI-AdS black hole as,

$$\begin{aligned} T &= \frac{1}{4\pi}\left(\frac{\partial f(r)}{\partial r}\right)_{r_+} \\ &= \frac{10kr_+^4 + 5k\alpha r_+^2 + 2b^2 r_+^6(1 - \sqrt{1+\eta_+}) - \Lambda r_+^6}{10\pi r_+(r_+^2 + k\alpha)^2} \end{aligned} \tag{14}$$

In the limit $\alpha \to 0$ the corresponding expression for the Hawking temperature of the Born-Infeld AdS (BI-AdS) black hole can be recovered as,

$$T_{BI-AdS} = \frac{1}{4\pi}\left[\frac{4k}{r_+} + \frac{4b^2 r_+}{5}\left(1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}}\right) - \frac{2\Lambda r_+}{5}\right]. \tag{15}$$

From the first law of black hole thermodynamics $dM = TdS + \Phi dQ$, we may obtain the entropy of the third order LBI-AdS black hole as,

$$\begin{aligned} S &= \int_0^{r_+} \frac{1}{T}\left(\frac{\partial M}{\partial r_+}\right)_Q dr_+ \\ &= \frac{\pi^3}{4}\left(r_+^5 + \frac{10k\alpha r_+^3}{3} + 5k^2\alpha^2 r_+\right). \end{aligned} \tag{16}$$

From (16) we find that the entropy is not proportional to the one-fourth of the horizon area as in the case of the black holes in the Einstein gravity. This is the most striking feature of the higher curvature gravity, the usual area law of black hole[57] breaks down in these gravity theories. If we take the limit $\alpha \to 0$, we can recover the usual area law of black hole entropy in the BI-AdS black hole[8]

$$S = \frac{\pi^3}{4} r_+^5. \tag{17}$$

In our study of critical phenomena we will be mainly concerned about the spherically symmetric space-time. In this regard we will always take the value of k to be $+1$. Substituting $k = 1$ in (14) and (16) we finally obtain the expressions for the Hawking temperature and the entropy of the third order LBI-AdS black hole as,

$$T = \frac{10r_+^4 + 5\alpha r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6}{10\pi r_+(r_+^2 + \alpha)^2} \tag{18}$$

and

$$S = \frac{\pi^3}{4}\left(r_+^5 + \frac{10\alpha r_+^3}{3} + 5\alpha^2 r_+\right). \tag{19}$$

3 Phase transition and stability of the third order LBI-AdS black hole

In this section we aim to discuss the nature of phase transition and the stability in the third order LBI-AdS black hole. The nature of phase transition of a thermodynamic system is usually determined by inspecting into the discontinuity in the relevant thermodynamic quantities at the critical point[24]. While the first order phase transition is characterized by the discontinuity in the temperature of the system, any discontinuity in the specific heat gives rise to a higher order phase transition. Since black holes are considered as thermodynamic objects one may also expect phase transition in these systems. Earlier research works revealed that phase transitions indeed occur in black holes[7, 8], [10]-[22]. In order to determine the order of phase transition in black holes we generally use the Ehrenfest's equations for black holes where the usual thermodynamic variables are replaced by those for the black holes[18]. We can further explore the stability of the third order LBI-AdS black hole by analyzing the specific heat of the black hole. The negative value of specific heat gives rise to the unstable configuration of the black hole whereas the positive value of the specific heat makes the black hole stable. We can further clarify the stability analysis of the black hole by discussing the nature of the free energy of the black hole.

Let us now plot the Hawking temperature (T) of the third order LBI-AdS black hole against the radius of the outer horizon (r_+) of the black hole.

We observe from the above figures that, there is a 'hump' and a 'dip' in the $T - r_+$ graph. Also it is very interesting to note that, the graphs are continuous in r_+. Since the entropy (S) of the black hole increases with the radius of the outer horizon (r_+), we may argue that the entropy of the third order LBI-AdS black hole is also a continuous function. This rules out the possibility

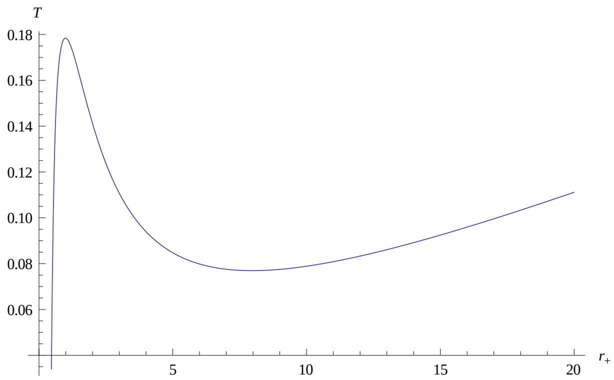

Figure 1: *Plot of Hawking temperature (T) against horizon radius (r_+), for $\alpha = 0.5$, $Q =$ 0.50 and $b = 10$.*

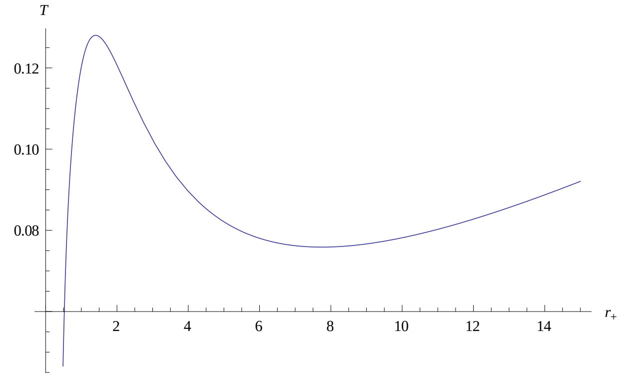

Figure 2: *Plot of Hawking temperature (T) against horizon radius (r_+), for $\alpha = 1.0$, $Q =$ 0.50 and $b = 10$.*

of first order phase transition. In our subsequent analysis we will see that the specific heat at constant charge (C_Q) shows an infinite discontinuity at the critical points which correspond to the 'hump' and 'dip' of the $T-r_+$ plots respectively. Without going into the algebraic details, we can argue, from our previous work[22], that this corresponds to a continuous higher order phase transition. In order to study the critical phenomena in the canonical ensemble framework, we need to calculate the specific heat at constant charge (C_Q). Using the standard thermodynamic definition we may write the specific heat at constant charge of the third order LBI-AdS black hole as,

$$\begin{aligned} C_Q &= T\left(\frac{\partial S}{\partial T}\right)_Q \\ &= T\frac{(\partial S/\partial r_+)_Q}{(\partial T/\partial r_+)_Q} \\ &= \frac{\mathcal{N}(r_+,Q)}{\mathcal{D}(r_+,Q)} \end{aligned} \tag{20}$$

where

$$\mathcal{N}(r_+,Q) = \frac{5}{4}\pi^7 r_+^5 (r_+^2+\alpha)^3 \left[10r_+^2 + 5\alpha + 2b^2 r_+^4\left(1-\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^4\right] \tag{21}$$

and

$$\begin{aligned} \mathcal{D}(r_+,Q) = 128Q^2 + \left[15\pi^4 r_+^6\alpha + 5\pi^4 r_+^4\alpha^2 - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8(2+\alpha\Lambda)\right] \\ \sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \left(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8\alpha\right)\left(1-\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) \end{aligned} \tag{22}$$

where we have used (18) and (19).

The analytic solution of the denominator $\mathcal{D}(r_+,Q)$ of (20) is very much problematic for arbitrary choices of the parameters α, b and Q. In order to see whether there is any bound in the values of b and Q we will consider the extremal third order LBI-AdS black hole. In this case both $f(r)$ and $\frac{df}{dr}$ vanish at the degenerate horizon r_e. The above two conditions for extremality results the following equation,

$$10r_e^4 + 5\alpha r_e^2 + 2b^2 r_e^6\left(1-\sqrt{1+\frac{16Q^2}{b^2\pi^4 r_e^{10}}}\right) - \Lambda r_e^6 = 0 \tag{23}$$

Table 1a: Roots of equation (23) for $\alpha = 0.5$ and $l = 10$

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	-66.4157	-	-	+0.632734
8	0.2	-66.4157	-	-	+0.121590
5	0.5	-66.4157	-	-	+0.200244
0.8	20	-66.4157	-0.408088	-0.05881$\pm i$0.304681	+0.268514
0.3	15	-66.4157	-0.304225	-0.0517567$\pm i$0.175488	+0.145685
0.5	10	-66.4157	-0.349764	-0.0593449$\pm i$0.240144	+0.192519
0.5	1	-66.4157	-	-	+0.0221585
0.5	0.5	-66.4157	-	-	+0.00625335
0.05	0.05	-66.4157	-	-	$+6.57019\times10^{-7}$

Table 1b: Roots of equation (23) for $\alpha = 1.0$ and $l = 10$

Q	b	r_{e1}^2	r_{e2}^2	$r_{e3,e4}^2$	r_{e5}^2
15	0.6	-66.1628	-	-	+0.50473
8	0.2	-66.1628	-	-	+0.0546589
5	0.5	-66.1628	-	-	+0.110054
0.8	20	-66.1629	-0.563564	-0.0913395$\pm i$0.266887	+0.236186
0.3	15	-66.1629	-0.514815	-0.0584676$\pm i$0.14609	+0.116834
0.5	10	-66.1629	-0.531635	-0.0760937$\pm i$0.210702	+0.155024
0.5	1	-66.1629	-0.542417	-	+0.00640486
0.5	0.5	-66.1629	-	-	+0.00163189
0.05	0.05	-66.1629	-	-	$+1.64256\times10^{-7}$

In Table 1a and 1b we give the numerical solution of (23) for different choices of the values of the parameters b and Q for fixed values of α. From these analysis we observe that for arbitrary choices of the parameters b and Q we always get atleast one real positive root of (23). This implies that there exists a smooth extremal limit for arbitrary b and Q and there is no bound on the parameter space for a particular value of α. This result is indeed expected since it was observed that, although, there is a bound on the parameters b and Q for four dimensional ($n = 3$) Born-Infeld AdS black holes, the bound is removed if we consider these black holes in higher dimensions ($n > 3$)[8, 44].

We shall now find out the roots of (22) for different values of the parameters b, Q and α. The numerical results are given in the Tables 2a and 2b below. For convenience we shall write the real roots of (22) only. From our analysis it is worth mentioning that the specific heat at constant charge of the third order LBI-AdS black hole always possesses simple poles. Also, there are two real positive roots (r_1, r_2) of the denominator of C_Q for different values of the parameters b, Q and α. Moreover, these two roots (r_1 and r_2) correspond exactly to the 'hump' and the 'dip' in the $T - r_+$ plot. Another interesting point can be observed from the $C_Q - r_+$ plots that C_Q shows an infinite discontinuity at the points r_1 and r_2, which may be identified as the critical points of phase transition in the third order LBI-AdS black hole. From the nature of the $T - r_+$ and $C_Q - r_+$ plots it is indeed obvious that the phase transition in the third order LBI-AdS black hole is not a first order phase transition, rather, the discontinuity of C_Q at the critical points (r_1 and r_2) reveals that it is a continuous higher order (i.e. greater than first) phase transition.

We can further analyze the stability of the third order LBI-AdS black hole once we study the behavior of C_Q at the critical points. The $C_Q - r_+$ plot shows that there are indeed three phases of the black hole. These phases can be classified as - Phase I ($0 < r_+ < r_1$), Phase

Table 2a: Real roots of equation (22) for $\alpha = 0.5$ and $l = 10$

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.59399	7.96087	-7.96087	-1.59399
8	0.2	1.18048	7.96088	-7.96088	-1.18048
5	0.5	1.25190	7.96088	-7.96088	-1.25190
0.8	20	1.01695	7.96088	-7.96088	-1.01695
0.3	15	0.975037	7.96088	-7.96088	-0.975037
0.5	10	0.989576	7.96088	-7.96088	-0.989576
0.5	1	0.989049	7.96088	-7.96088	-0.989049
0.5	0.5	0.987609	7.96088	-7.96088	-0.987609
0.05	0.05	0.965701	7.96088	-7.96088	-0.965701

Table 2b: Real roots of equation (22) for $\alpha = 1.0$ and $l = 10$

Q	b	r_1	r_2	r_3	r_4
15	0.6	1.76824	7.74534	-7.74534	-1.76824
8	0.2	1.53178	7.74535	-7.74535	-1.53178
5	0.5	1.51668	7.74535	-7.74535	-1.51668
0.8	20	1.40553	7.74535	-7.74535	-1.40553
0.3	15	1.40071	7.74535	-7.74535	-1.40071
0.5	10	1.40214	7.74535	-7.74535	-1.40214
0.5	1	1.40214	7.74535	-7.74535	-1.40214
0.5	0.5	1.40213	7.74535	-7.74535	-1.40213
0.05	0.05	1.39992	7.74535	-7.74535	-1.39992

II ($r_1 < r_+ < r_2$) and Phase III ($r_+ > r_2$). Since the higher mass black hole possesses larger entropy/horizon radius, there is a phase transition at r_1 from smaller mass black hole (Phase I) to intermediate (higher mass) black hole (Phase II). The critical point r_2 corresponds to a phase transition from an intermediate (higher mass) black hole (Phase II) to a larger mass black hole (Phase III). Moreover, from the $C_Q - r_+$ plots we note that the specific heat C_Q is positive for Phase I and Phase III whereas it is negative for Phase II. Therefore Phase I and Phase III correspond to thermodynamically stable phase ($C_Q > 0$) whereas Phase II corresponds to thermodynamically unstable phase ($C_Q < 0$).

We can further extend our stability analysis by considering the free energy of the third order LBI-AdS black hole. The free energy plays an important role in the theory of phase transition and critical phenomena. We may define the free energy of the third order LBI-AdS black hole as,

$$\mathcal{F}(r_+, Q) = M(r_+, Q) - TS \tag{24}$$

where $M(r_+, Q)$, T and S are the ADM mass, the Hawking temperature and the entropy of the black hole respectively.

Using (11), (18) and (19) we can write (24) as

$$\mathcal{F} = \frac{5\pi^2}{16}\left[\frac{\alpha^2}{3} + r_+^4 + \alpha r_+^2 + \frac{2b^2 r_+^6}{15}\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}} - \frac{\Lambda}{2b^2} + \frac{20Q^2}{b^2\pi^4 r_+^{10}}\mathcal{H}\left(\frac{1}{2}, \frac{2}{5}, \frac{7}{5}, -\frac{16Q^2}{b^2\pi^4 r_+^{10}}\right)\right)\right]$$
$$-\frac{\pi^2(r_+^5 + \frac{10\alpha r_+^3}{3} + 5\alpha^2 r_+)}{40 r_+(r_+ + \alpha)^2}\left[10r_+^4 + 5\alpha r_+^2 + 2b^2 r_+^6\left(1 - \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_+^{10}}}\right) - \Lambda r_+^6\right]. \tag{25}$$

We plot the free energy ($\mathcal{F}$) of the black hole with the radius of the outer horizon r_+ in the

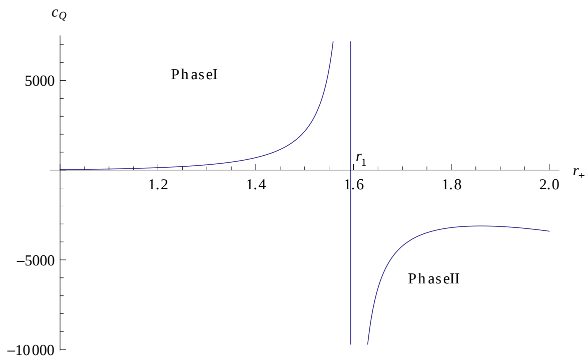

Figure 3: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 0.5$, $Q = 15$ and $b = 0.60$ at the critical point r_1.*

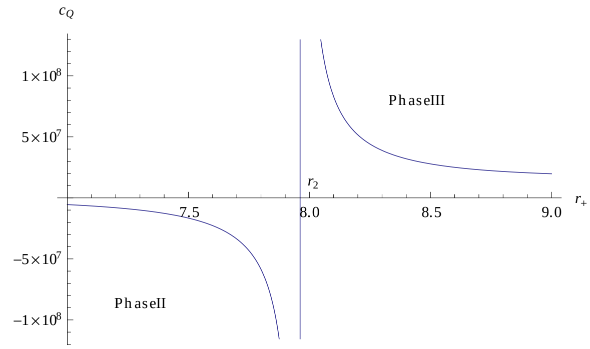

Figure 4: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 0.5$, $Q = 15$ and $b = 0.60$ at the critical point r_2.*

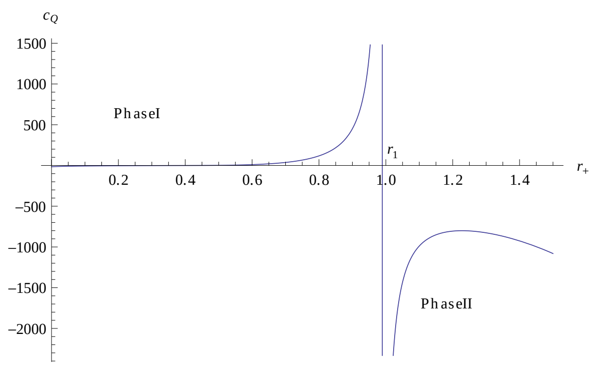

Figure 5: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 0.5$, $Q = 0.50$ and $b = 10$ at the critical point r_1.*

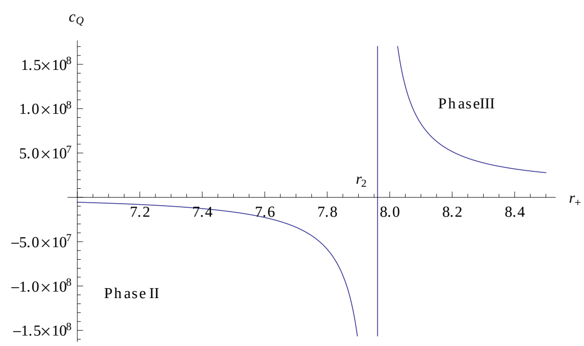

Figure 6: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 0.5$, $Q = 0.50$ and $b = 10$ at the critical point r_2.*

following figures (Fig. 11-14).

Let us now analyze the $\mathcal{F} - r_+$ plots. The free energy ($\mathcal{F}$) has a minima $\mathcal{F} = \mathcal{F}_m$ at $r_+ = r_m$. This point of minimum free energy is exactly the same as the first critical point $r_+ = r_1$ where the black hole shifts from a stable to an unstable phase. On the other hand $\mathcal{F}$ has a maxima $\mathcal{F} = \mathcal{F}_0$ at $r_+ = r_0$. The point at which $\mathcal{F}$ reaches its maximum value, is identical with the second critical point $r_+ = r_2$ where the black hole changes from unstable to stable phase. We can further divide the $\mathcal{F} - r_+$ plot into three distinct regions. In the first region $r_1' < r_+ < r_m$ the negative free energy decreases until it reaches the minimum value ($\mathcal{F}_m$) at $r_+ = r_m$. This region corresponds to the stable phase (Phase I: $C_Q > 0$) of the black hole. The free energy changes its slope at $r_+ = r_m$ and continues to increase in the second region $r_m < r_+ < r_0$ approaching towards the maximum value ($\mathcal{F}_0$) at $r_+ = r_0$. This region corresponds to the Phase II of the $C_Q - r_+$ plot where the black hole becomes unstable ($C_Q < 0$). The free energy changes its slope once again at $r_+ = r_0$ and decreases to zero at $r_+ = r_2'$ and finally becomes negative for $r_+ > r_2'$. This region of the $\mathcal{F} - r_+$ plot corresponds to the Phase III of the $C_Q - r_+$ plot where the black hole finally becomes stable ($C_Q > 0$).

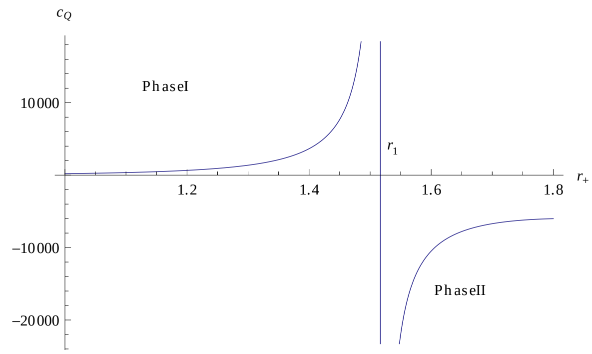

Figure 7: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 5$ and $b = 0.5$ at the critical point r_1.*

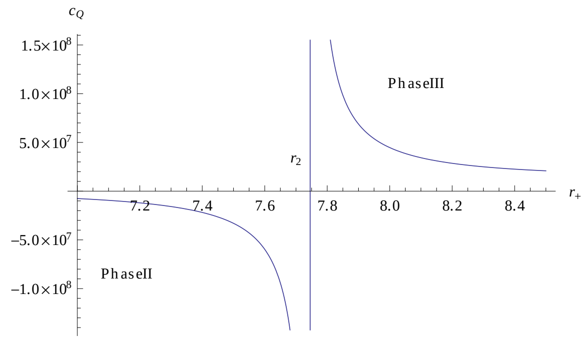

Figure 8: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 5$ and $b = 0.5$ at the critical point r_2.*

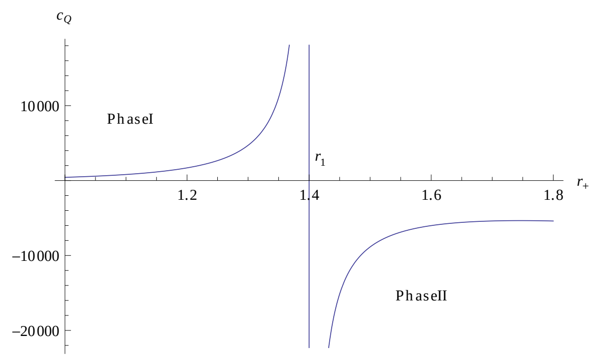

Figure 9: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 0.05$ and $b = 0.05$ at the critical point r_1.*

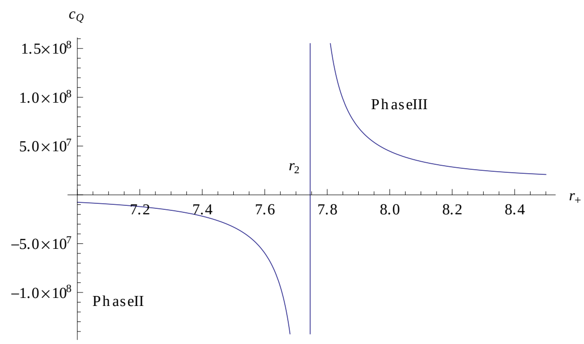

Figure 10: *Plot of specific heat (C_Q) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 0.05$ and $b = 0.05$ at the critical point r_2.*

4 Critical exponents and scaling hypothesis

In thermodynamics, the theory of phase transition plays a crucial role to understand the behavior of a thermodynamic system. The behavior of thermodynamic quantities near the critical point(s) of phase transition gives a considerable amount of information about the system. The behavior of a thermodynamic system near the critical point(s) is usually studied by means of a set of indices known as the critical exponents[26, 27]. These are generally denoted by a set of Greek letters: α, β, γ, δ, ϕ, ψ, η, ν. The critical exponents describe the nature of singularities in various measurable thermodynamic quantities near the critical point(s).

In this section we aim to determine the first six static critical exponents (α, β, γ, δ, ϕ, ψ). We shall then discuss about the static scaling laws and static scaling hypothesis. We shall determine the other two critical exponents (ν and η) from two additional scaling laws.

Critical exponent α***:***

In order to determine the critical exponent α which is associated with the singularity of C_Q near the critical points r_i ($i = 1, 2$), we choose a point in the infinitesimal neighborhood of r_i as,

$$r_+ = r_i(1 + \Delta), \quad i = 1, 2 \tag{26}$$

where $|\Delta| << 1$. Let us denote the temperature at the critical point by $T(r_i)$ and define the

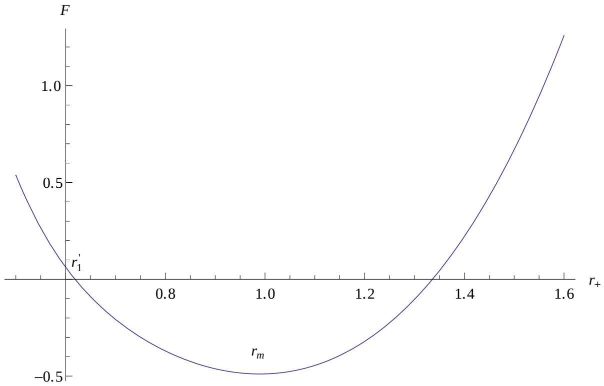

Figure 11: *Plot of free energy($\mathcal{F}$) against horizon radius (r_+), for $\alpha = 0.50$, $Q = 0.50$ and $b = 10$.*

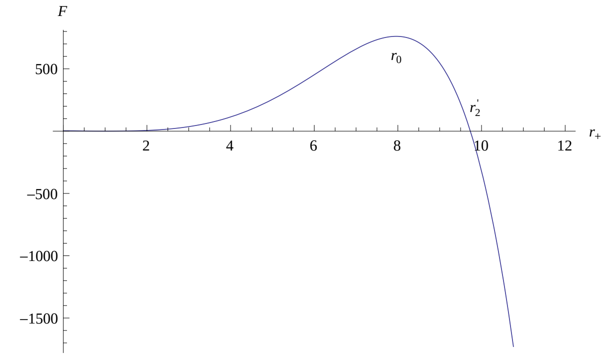

Figure 12: *Plot of free energy($\mathcal{F}$) against horizon radius (r_+), for $\alpha = 0.50$, $Q = 0.50$ and $b = 10$.*

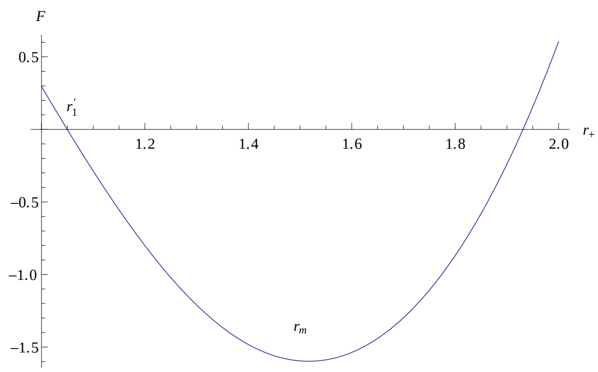

Figure 13: *Plot of free energy($\mathcal{F}$) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 5$ and $b = 0.5$.*

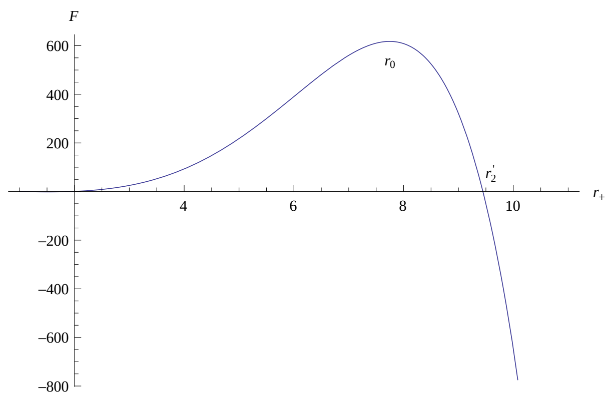

Figure 14: *Plot of free energy($\mathcal{F}$) against horizon radius (r_+), for $\alpha = 1.0$, $Q = 5$ and $b = 0.5$.*

quantity

$$\epsilon = \frac{T(r_+) - T(r_i)}{T(r_i)} \tag{27}$$

such that $|\epsilon| << 1$.

We now Taylor expand $T(r_+)$ in the neighborhood of r_i keeping the charge constant ($Q = Q_c$), which yields,

$$T(r_+) = T(r_i) + \left[\left(\frac{\partial T}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i)^2 + ... \tag{28}$$

Since the divergence of C_Q results from the vanishing of $\left(\frac{\partial T}{\partial r_+}\right)_Q$ at the critical point r_i ((20)), we may write (28) as,

$$T(r_+) = T(r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 T}{\partial r_+^2}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i)^2 \tag{29}$$

where we have neglected the higher order terms in (28).

Using (26) and (27) we can finally write (29) as,

$$\Delta = \frac{\epsilon^{1/2}}{\Gamma_i^{1/2}} \tag{30}$$

where

$$\Gamma_i = \frac{r_i^2}{2T(r_i)} \left[\left(\frac{\partial^2 T}{\partial r_+^2} \right)_{Q=Q_c} \right]_{r_+=r_i} \tag{31}$$

The detailed expression of Γ_i is very much cumbersome and we shall not write it down for the present work.

If we examine the $T - r_+$ plots (Fig. 1 and 2) we observe that, near the critical point $r_+ = r_1$ (which corresponds to the 'hump') $T(r_+) < T(r_1)$ so that $\epsilon < 0$ and on the contrary, near the critical point $r_+ = r_2$ (which corresponds to the 'dip') $T(r_+) > T(r_2)$ implying $\epsilon > 0$.

Substituting (26) into (20) we can write the singular part of C_Q as,

$$C_Q = \frac{\mathcal{N}'(r_i, Q_c)}{\Delta \cdot \mathcal{D}'(r_i, Q_c)} \tag{32}$$

where $\mathcal{N}'(r_i, Q_c)$ is the value of the numerator of C_Q ((21)) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$. The expression for $\mathcal{D}'(r_i, Q_c)$ is given by,

$$\mathcal{D}'(r_i, Q_c) = \mathcal{D}_1'(r_i, Q_c) + \mathcal{D}_2'(r_i, Q_c) + \mathcal{D}_3'(r_i, Q_c) \tag{33}$$

where

$$\mathcal{D}_1'(r_i, Q_c) = 10\pi^4 r_i^4 \sqrt{1 + \frac{16Q^2}{b^2\pi^4 r_i^{10}}} \left[(2\alpha^2 + 2b^2 r_i^6 + 8b^2 r_i^4 \alpha) - (\Lambda r_i^6 + 9\alpha r_i^2 + 4(2+\Lambda\alpha) r_i^4) \right]$$

$$\mathcal{D}_2'(r_i, Q_c) = -\frac{80Q^2}{b^2 r_i^2} \left[\frac{15\alpha}{r_i^2} + \frac{5\alpha^2}{r_i^4} - \Lambda r_i^2 - 5(2+\Lambda\alpha) \right]$$

$$\mathcal{D}_3'(r_i, Q_c) = 20 b^2 \pi^4 r_i^8 \left[r_i^2 \left(1 + \frac{8Q^2}{b^2\pi^4 r_i^{10}} \right) + 4\alpha \left(1 + \frac{10Q^2}{b^2\pi^4 r_i^{10}} \right) \right]$$

It is to be noted that while expanding the denominator of C_Q we have retained the terms which are linear in Δ and all other higher order terms of Δ have been neglected.

Using (32) we may summarize the critical behavior of C_Q near the critical points (r_1 and r_2) as follows:

$$C_Q \sim \begin{cases} \left[\frac{\mathcal{A}_i}{(-\epsilon)^{1/2}} \right]_{r_i=r_1} & \epsilon < 0 \\ \left[\frac{\mathcal{A}_i}{(+\epsilon)^{1/2}} \right]_{r_i=r_2} & \epsilon > 0 \end{cases} \tag{34}$$

where

$$\mathcal{A}_i = \frac{\Gamma_i^{1/2} \mathcal{N}'(r_i, Q_c)}{\mathcal{D}'(r_i, Q_c)} \tag{35}$$

We can combine the r.h.s of (34) into a single expression, which describes the singular nature of C_Q near the critical point r_i, yielding

$$\begin{aligned} C_Q &= \frac{\mathcal{A}_i}{|\epsilon|^{1/2}} \\ &= \frac{\mathcal{A}_i T_i^{1/2}}{|T - T_i|^{1/2}}. \end{aligned} \tag{36}$$

where we have used(27). Here we have abbreviated T and T_i for $T(r_+)$ and $T(r_i)$ respectively.

We can now compare (36) with the standard form

$$C_Q \sim |T - T_i|^{-\alpha} \tag{37}$$

which gives $\alpha = \frac{1}{2}$.

Critical exponent β **:**

The critical exponent β is related to the electric potential at infinity (Φ) by the relation

$$\Phi(r_+) - \Phi(r_i) \sim |T - T_i|^{\beta} \tag{38}$$

where the charge (Q) is kept constant.

Near the critical point $r_+ = r_i$ the Taylor expansion of $\Phi(r_+)$ yields,

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i} (r_+ - r_i) + ... \tag{39}$$

Neglecting the higher order terms and using (12) and (30) we can rewrite (39) as,

$$\Phi(r_+) - \Phi(r_i) = -\left(\frac{4Q_c}{\pi^2 r_i^4 \Gamma_i^{1/2} T_i^{1/2}\sqrt{1 + \frac{16Q_c^2}{b^2\pi^4 r_i^{10}}}}\right)|T - T_i|^{1/2} \tag{40}$$

Comparing (40) with (38) we finally obtain $\beta = \frac{1}{2}$.

Critical exponent γ:

We shall now determine the critical exponent γ which is associated with the singularity of the inverse of the isothermal compressibility (κ_T^{-1}) at constant charge $Q = Q_c$ near the critical point $r_+ = r_i$ as,

$$\kappa_T^{-1} \sim |T - T_i|^{-\gamma} \tag{41}$$

In order to calculate κ_T^{-1} we use the standard thermodynamic definition

$$\begin{aligned} \kappa_T^{-1} &= Q\left(\frac{\partial \Phi}{\partial Q}\right)_T \\ &= -Q\left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi \end{aligned} \tag{42}$$

where in the last line of (42) we have used the identity

$$\left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi \left(\frac{\partial Q}{\partial \Phi}\right)_T = -1 \tag{43}$$

Using (12) and (14) we can write (42) as,

$$\kappa_T^{-1} = \frac{\Omega(r_+, Q)}{\mathcal{D}(r_+, Q)} \tag{44}$$

where $\mathcal{D}(r_+, Q)$ is the denominator identically equal to (22) (the denominator of C_Q) and the expression for $\Omega(r_+, Q)$ may be written as,

$$\Omega(r_+, Q) = \frac{Q}{5\pi^2 r_+^4}[128Q^2 + (15\pi^4 r_+^6\alpha + 5\pi^4 r_+^4\alpha^2 - \Lambda\pi^4 r_+^{10} - 5\pi^4 r_+^8(2+\alpha\Lambda))\sqrt{1+(16Q^2/b^2\pi^4 r_+^{10})}$$
$$-(2b^2\pi^4 r_+^{10} + 10b^2\pi^4 r_+^8\alpha)\left(1-\sqrt{1+(16Q^2/b^2\pi^4 r_+^{10})}\right)\mathcal{H}(2/5, 1/2, 7/5, -16Q^2/b^2\pi^4 r_+^{10})] - \Sigma(r_+, Q) \quad (45)$$

where

$$\Sigma(r_+, Q) = \frac{4Q}{5}[2b^2\pi^2 r_+^4(r_+^2 + 5\alpha)\sqrt{1+(16Q^2/b^2\pi^4 r_+^{10})} +$$
$$\pi^2(-15r_+^2\alpha - 5\alpha^2 + r_+^6(\Lambda - 2b^2) + 5r_+^4(2+\Lambda\alpha - 2b^2\alpha))]$$

From (44) we observe that κ_T^{-1} possesses simple poles. Moreover κ_T^{-1} and C_Q exhibit common singularities.

We are now interested in the behavior of κ_T^{-1} near the critical point $r_+ = r_i$. In order to do so we substitute (26) into (44). The resulting equation for the singular part of κ_T^{-1} may be written as,

$$\kappa_T^{-1} = \frac{\Omega'(r_i, Q_c)}{\Delta \cdot \mathcal{D}'(r_i, Q_c)}. \quad (46)$$

In (46), $\Omega'(r_i, Q_c)$ is the value of the numerator of κ_T^{-1} ((45)) at the critical point $r_+ = r_i$ and critical charge $Q = Q_c$. Whereas $\mathcal{D}'(r_i, Q_c)$ was identified earlier ((33)).

Substituting (30) in (46) we may express the singular nature of κ_T^{-1} near the critical points (r_1 and r_2) as,

$$\kappa_T^{-1} \simeq \begin{cases} \left[\frac{\mathcal{B}_i}{(-\epsilon)^{1/2}}\right]_{r_i=r_1} & \epsilon < 0 \\ \left[\frac{\mathcal{B}_i}{(+\epsilon)^{1/2}}\right]_{r_i=r_2} & \epsilon > 0 \end{cases} \quad (47)$$

where

$$\mathcal{B}_i = \frac{\Gamma_i^{1/2}\Omega'(r_i, Q_c)}{\mathcal{D}'(r_i, Q_c)} \quad (48)$$

Combining the r.h.s of (47) into a single expression as before, we can express the singular behavior of κ_T^{-1} near the critical point r_i as,

$$\begin{aligned} \kappa_T^{-1} &= \frac{\mathcal{B}_i}{|\epsilon|^{1/2}} \\ &= \frac{\mathcal{B}_i T_i^{1/2}}{|T - T_i|^{1/2}}. \end{aligned} \quad (49)$$

Comparing (49) with (41) we find $\gamma = \dfrac{1}{2}$.

Critical exponent δ:

Let us now calculate the critical exponent δ which is associated with the electrostatic potential (Φ) for the fixed value $T = T_i$ of temperature. The relation can be written as,

$$\Phi(r_+) - \Phi(r_i) \sim |Q - Q_i|^{1/\delta} \quad (50)$$

in this relation Q_i is the value of charge (Q) at the critical point r_i. In order to obtain δ we first Taylor expand $Q(r_+)$ around the critical point $r_+ = r_i$. This yields,

$$Q(r_+) = Q(r_i) + \left[\left(\frac{\partial Q}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i}(r_+ - r_i) + \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_{T=T_i}\right]_{r_+=r_i}(r_+ - r_i)^2 + ... \quad (51)$$

Neglecting the higher order terms we can write (51) as,

$$Q(r_+) - Q(r_i) = \frac{1}{2}\left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i}(r_+ - r_i)^2 \quad (52)$$

Here we have used the standard thermodynamic identity

$$\left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left[\left(\frac{\partial r_+}{\partial T}\right)_Q\right]_{r_+=r_i}\left(\frac{\partial T}{\partial Q}\right)_{r_+=r_i} = -1 \quad (53)$$

and considered the fact that at the critical point $r_+ = r_i$, $\left(\frac{\partial T}{\partial r_+}\right)_Q$ vanishes.

Let us now define a quantity

$$\Upsilon = \frac{Q(r_+) - Q_i}{Q_i} = \frac{Q - Q_i}{Q_i} \quad (54)$$

where $|\Upsilon| << 1$. Here we denote $Q(r_+)$ and $Q(r_i)$ by Q and Q_i respectively.

Using (26) and (54) from (52) we obtain

$$\Delta = \frac{\Upsilon^{1/2}}{\Psi_i^{1/2}}\left[\frac{2Q_i}{r_i^2}\right]^{1/2} \quad (55)$$

where

$$\Psi_i = \left[\left(\frac{\partial^2 Q}{\partial r_+^2}\right)_T\right]_{r_+=r_i} \quad (56)$$

The expression for Ψ_i is very much cumbersome and we shall not write it in this paper.

We shall now consider the functional relation

$$\Phi = \Phi(r_+, Q) \quad (57)$$

from which we may write,

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_T\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_i} + \left[\left(\frac{\partial Q}{\partial r_+}\right)_T\right]_{r_+=r_i}\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+=r_i} \quad (58)$$

Using (53) we can rewrite (58) as,

$$\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i} = \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{Q=Q_c}\right]_{r_+=r_i} \quad (59)$$

Now the Taylor expansion of Φ at constant temperature around $r_+ = r_i$ yields,

$$\Phi(r_+) = \Phi(r_i) + \left[\left(\frac{\partial \Phi}{\partial r_+}\right)_{T=T_i}\right]_{r_+=r_i}(r_+ - r_i) \quad (60)$$

where we have neglected all the higher order terms.

Finally using (55), (59) and (12) we may write (60) as,

$$\Phi(r_+) - \Phi(r_i) = \left(\frac{-4Q_c}{\pi^2 r_i^5 \sqrt{1 + \frac{16Q_c^2}{b^2\pi^4 r_i^{10}}}} \right) \left(\frac{2}{\Psi_i} \right)^{\frac{1}{2}} |Q - Q_i|^{\frac{1}{2}} \tag{61}$$

Comparing (50) and (61) we find that $\delta = 2$.

Critical exponent ϕ:

The critical exponent ϕ is associated with the divergence of the specific heat at constant charge (C_Q) at the critical point $r_+ = r_i$ as,

$$C_Q \sim |Q - Q_i|^{-\phi}. \tag{62}$$

Now from (32) and (36) we note that,

$$C_Q \sim \frac{1}{\Delta} \tag{63}$$

which may be written as,

$$C_Q \sim \frac{1}{|Q - Q_i|^{1/2}} \tag{64}$$

where we have used Equation (55).

Comparison of (64) with (62) yields $\phi = \dfrac{1}{2}$.

Critical exponent ψ:

In order to calculate the critical exponent ψ, which is related to the entropy of the third order LBI-AdS black hole, we Taylor expand the entropy ($S(r_+)$) around the critical point $r_+ = r_i$. This gives,

$$S(r_+) = S(r_i) + \left[\left(\frac{\partial S}{\partial r_+} \right) \right]_{r_+ = r_i} (r_+ - r_i) + ... \tag{65}$$

If we neglect all the higher order terms and use (19), (26) and (55), we can write (65) as,

$$S(r_+) - S(r_i) = \frac{5\pi^3}{4} \left(r_i^4 + 2\alpha r_i^2 + \alpha^2 \right) \left(\frac{2}{\Psi_i} \right)^{1/2} |Q - Q_i|^{1/2} \tag{66}$$

Comparing (66) with the standard relation

$$S(r_+) - S(r_i) \sim |Q - Q_c|^{\psi} \tag{67}$$

we finally obtain $\psi = \dfrac{1}{2}$.

In the table below we write all the six critical exponents obtained from our analysis in a tabular form. For comparison we also give the critical exponents associated with some well known systems.

Critical exponents	3rd order LBI-AdS black hole	CrBr_3 *	2D Ising model*	van der Waals's system‡
α	0.5	0.05	0	0
β	0.5	0.368	0.125	0.5
γ	0.5	1.215	1.7	1.0
δ	2.0	4.28	15	3.0
ψ	0.5	0.60	-	-
ϕ	0.5	0.03	-	-

(*: these are the non-mean field values.)
(‡: these values are taken from [36].)

Thermodynamic scaling laws and static scaling hypothesis:

The discussion on critical phenomena is far from complete unless we make a comment on the thermodynamic scaling laws. In standard thermodynamic systems the critical exponents are found to satisfy some relations among themselves. These relations are called thermodynamic scaling laws[26, 27]. These scaling relations are given as,

$$\begin{aligned}
\alpha + 2\beta + \gamma &= 2 \\
\alpha + \beta(\delta + 1) &= 2 \\
\phi + 2\psi - \frac{1}{\delta} &= 1 \\
\beta(\delta - 1) &= \gamma \\
(2 - \alpha)(\delta - 1) &= \gamma(1 + \delta) \\
1 + (2 - \alpha)(\delta\psi - 1) &= (1 - \alpha)\delta
\end{aligned} \tag{68}$$

From the values of the critical exponents obtained in our analysis it is interesting to observe that all these scaling relations are indeed satisfied for the third order LBI-AdS black holes.

We shall now explore the static scaling hypothesis[26]-[28] for the third order LBI-AdS black hole. Since we are working in the canonical framework, the thermodynamic potential of interest is the *Helmhotz free energy*, $\mathcal{F}(T, Q) = M - TS$, where the symbols have their usual meaning.

Now the static scaling hypothesis states that,

Close to the critical point the singular part of the Helmhotz free energy is a generalized homogeneous function of its variables.

This asserts that there exist two parameters a_ϵ and a_Υ such that

$$\mathcal{F}(\lambda^{a_\epsilon}\epsilon, \lambda^{a_\Upsilon}\Upsilon) = \lambda\mathcal{F}(\epsilon, \Upsilon) \tag{69}$$

for any arbitrary number λ.

In an attempt to find the values of the scaling parameters a_ϵ and a_Υ, we shall now Taylor expand the Helmhotz free energy $\mathcal{F}(T, Q)$ near the critical point $r_+ = r_i$. This may be written as,

$$\begin{aligned}
\mathcal{F}(T, Q) = \mathcal{F}(T, Q)|_{r_+=r_i} &+ \left[\left(\frac{\partial \mathcal{F}}{\partial T}\right)_Q\right]_{r_+=r_i} (T - T_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i} (T - T_i)^2 \\
&+ \left[\left(\frac{\partial \mathcal{F}}{\partial Q}\right)_T\right]_{r_+=r_i} (Q - Q_i) + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i} (Q - Q_i)^2 \\
&+ \left[\left(\frac{\partial^2 \mathcal{F}}{\partial T \partial Q}\right)\right]_{r_+=r_i} (T - T_i)(Q - Q_i) + ...
\end{aligned} \tag{70}$$

From (70) we can identify the second derivatives of $\mathcal{F}$ as,

$$\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q = \frac{-C_Q}{T} \tag{71}$$

and

$$\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T = \frac{\kappa_T^{-1}}{Q} \tag{72}$$

Note that, since both C_Q and κ_T^{-1} diverge at the critical point, these derivatives can be justified as the singular parts of the free energy $\mathcal{F}$.

Since in the theory of critical phenomena we are mainly interested in the singular part of the relevant thermodynamic quantities, we sort out the singular part of $\mathcal{F}(T, Q)$ from (70), which may be written as,

$$\begin{aligned}
\mathcal{F}_s &= \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial T^2}\right)_Q\right]_{r_+=r_i}(T-T_i)^2 + \frac{1}{2}\left[\left(\frac{\partial^2 \mathcal{F}}{\partial Q^2}\right)_T\right]_{r_+=r_i}(Q-Q_i)^2 \\
&= \frac{-C_Q}{2T_i}(T-T_i)^2 + \frac{\kappa_T^{-1}}{2Q}(Q-Q_i)^2
\end{aligned} \tag{73}$$

where the subscript 's' denotes the singular part of the free energy $\mathcal{F}$.

Using (30), (36), (49) and (55) we may write the singular part of the Helmhotz free energy ($\mathcal{F}$) as,

$$\mathcal{F}_s = \sigma_i \epsilon^{3/2} + \tau_i \Upsilon^{3/2} \tag{74}$$

where

$$\sigma_i = \frac{-\mathcal{A}_i T_i}{2} \quad and, \quad \tau_i = \frac{\mathcal{B}_i \Psi_i^{1/2} Q_i^{1/2} r_i}{2^{3/2}\Gamma_i^{1/2}}. \tag{75}$$

From (69) and (74) we observe that

$$a_\epsilon = a_\Upsilon = \frac{2}{3} \tag{76}$$

This is an interesting result in the sense that, in general, a_ϵ and a_Υ are different for a generalized homogeneous function (GHF), but in this particular model of the black hole these two scaling parameters are indeed identical. With this result we can argue that, the Helmhotz free energy is an usual homogeneous function for the third order LBI-AdS black hole. Moreover, we can determine the critical exponents (α, β, γ, δ, ϕ, ψ) once we calculate the scaling parameters. This is because these critical exponents are related to the scaling parameters as[26, 27],

$$\begin{aligned}
\alpha &= 2 - \frac{1}{a_\epsilon} \\
\beta &= \frac{1-a_\Upsilon}{a_\epsilon} \\
\gamma &= \frac{2a_\Upsilon - 1}{a_\epsilon} \\
\delta &= \frac{a_\Upsilon}{1-a_\Upsilon} \\
\phi &= \frac{2a_\epsilon - 1}{a_\Upsilon} \\
\psi &= \frac{1-a_\epsilon}{a_\Upsilon}
\end{aligned} \tag{77}$$

There are two other critical exponents associated with the behavior of the *correlation function* and *correlation length* of the system near the critical surface. We shall denote these two critical exponents as η and ν respectively. If $G(\vec{r}_+)$ and ξ are the correlation function and the correlation length respectively, we can relate η and ν with them as

$$\xi \sim |T - T_i|^{-\nu} \tag{78}$$

and

$$G(\vec{r}_+) \sim r_+^{2-n-\eta}. \tag{79}$$

For the time being we shall assume that the two additional scaling relations[26]

$$\gamma = \nu(2-\eta) \quad and \quad (2-\alpha) = \nu n \tag{80}$$

hold for the third order LBI-AdS black hole. Using these two relations ((80)) and the values of α and γ, the exponents ν and η are found to be $\frac{1}{4}$ and 0 respectively.

Although we have calculated η and ν assuming the additional scaling relations to be valid, it is not proven yet that these scaling relations are indeed valid for the black holes. One may adapt different techniques to calculate η and ν, but till now no considerable amount of progress has been made in this direction. One may compute these two exponents directly from the correlation of scalar modes in the theory of gravitation[39] but the present theories of critical phenomena in black holes are far from complete.

5 Conclusion

In this paper we have analyzed the critical phenomena in higher curvature charged black holes in a canonical framework. For this purpose we have considered the third order Lovelock-Born-Infeld-AdS (LBI-AdS) black holes in a spherically symmetric space-time. We systematically derived the thermodynamic quantities for such black holes. We are able to show that some of the thermodynamic quantities (C_Q, κ_T^{-1}) diverge at the critical points. From the nature of the plots we argue that there is a higher order phase transition in this black hole. Although the analytical estimation of the critical points is not possible due to the complexity of the relevant equations, we are able to determine the critical points numerically. However, all the critical exponents are calculated analytically near the critical points. Unlike the AdS black holes in the Einstein gravity, one interesting property of the higher curvature black holes is that the usual area law of entropy does not hold for these black holes. One might then expect that the critical exponents may differ form those for the AdS black holes in the Einstein gravity. But we find that all the critical exponents in the third order LBI-AdS black hole are indeed identical with those obtained in Einstein gravity[8, 44]. From this observation we may conclude that these black holes belong to the same universality class. Moreover, the critical exponents take the mean field values. It is to be noted that these black holes have distinct set of critical exponents which does not match with the critical exponents of any other known thermodynamic systems. Another point that must be stressed is that the static critical exponents are independent of the spatial dimensionality of the AdS space-time. This suggests the mean field behavior in black holes as thermodynamic systems and allows us to study the phase transition phenomena in the black holes. We have also discussed the static scaling laws and static scaling hypothesis. The static critical exponents are found to satisfy the static scaling laws near the critical points. We have checked the consistency of the static scaling hypothesis. Apart from this we note that the two scaling parameters have identical values. This allows us to conclude that the Helmhotz free energy is indeed a homogeneous function for this type of black hole. We have determined the two other critical exponents ν and η associated with the correlation length (ξ) and correlation

function ($G(\vec{r}_+)$) near the critical surface assuming the validity of the additional scaling laws. The values of these two exponents are found to be $\frac{1}{4}$ and 0 respectively in the six spatial dimensions. Although the other six critical exponents are independent of the spatial dimension of the system, these two exponents are very much dimension dependent.

In our analysis we have been able to resolve a number of vexing issues concerning the critical phenomena in third order LBI-AdS black holes. But there still remains some unsolved problems that encourage one to make further investigations into the system. First of all, we have made a qualitative argument about the nature of the phase transition in this black hole. One needs to go through detailed algebraic analysis in order to determine the true order of the phase transition[18]-[22]. Secondly, we have calculated the values of the exponents ν and η assuming that the additional scaling relations hold for this black hole. But there is no evidence whether these two laws hold for the black hole[8, 44]. These scaling relations may or may not hold for the black hole. The dimension dependence of these two exponents (ν and η) makes the issue highly nontrivial in higher dimensions. A further attempt to determine η and ν may be based on Ruppeiner's prescription [58], where it is assumed that the absolute value of the thermodynamic scalar curvature ($|\mathcal{R}|$) is proportional to the correlation volume ξ^n:

$$|\mathcal{R}| \sim \xi^n \tag{81}$$

where n is the spatial dimension of the black hole. Now if we can calculate $\mathcal{R}$ using the standard method[22, 59, 60], we can easily determine ξ from (81). Evaluating ξ around the critical point $r_+ = r_i$ as before, we can determine ν directly. It is then straight forward to calculate η by using (80). This alternative approach, based on Ruppeiner's prescription, to determine ν and η needs high mathematical rigor and also the complexity in the determination of the scalar curvature ($\mathcal{R}$) in higher dimensions makes the issue even more challenging.

Finally, it would be highly nontrivial if we aim to investigate the AdS/CFT duality as an alternative approach to make further insight into the theory of critical phenomena in these black holes. The *renormalization group* method may be another alternative way to describe the critical phenomena in these black holes.

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